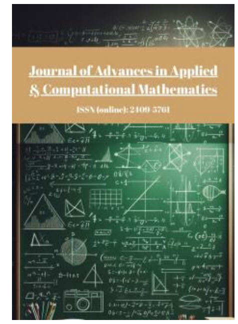
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



Conditional Randomized Divergence Degree of Theories in the Fuzzy Propositional Logic System BL^*

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ARTICLE INFO

Article Type: Research Article

Academic Editor: Wei Yao

Keywords:
Fuzzy logic
Conditional randomized truth degree
Conditional randomized divergence degree
Conditional randomized logical metric space

Timeline:
Received: January 13, 2026
Accepted: February 20, 2026
Published: March 07, 2026

Citation: He Y, Hui X, Peng C. Conditional randomized divergence degree of theories in the fuzzy propositional logic system BL*. J Adv Appl Computat Math. 2026; 13: 1-16.

DOI: https://doi.org/10.15377/2409-5761.2026.13.1

ABSTRACT

Using the concept of conditional probability and the randomization method of valuation sets, the conditional randomized truth degree of formulas is proposed in the fuzzy propositional logic system BL^* with valuation domain [0,1]. The MP rule and HS rule for conditional randomized truth degrees are proven based on the semantics of BR_0 algebra. At the same time, the concepts of conditional randomized similarity and conditional randomized pseudo-metric between formulas are introduced. It established a conditional randomized logic metric space. Within this conditional randomized logic metric space, the conditional randomized divergence of theories and its equivalent forms are proposed. Along with three different types of approximate reasoning modes, the relationships among them are discussed.

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1. Introduction

Mathematical logic is characterized by its core features of symbolization and formalization, focusing on formal reasoning and emphasizing the rigor of arguments. In contrast, computational mathematics centers on numerical computation and allows for approximate solutions. The integration of the symbolization and formalization of mathematical logic with the numerical computation and approximate solutions of computational mathematics represents a research direction pursued by numerous scholars. In the 1970s, Pavelka proposed a comprehensive graded logical theory within the framework of lattice-valued propositional logic [1], laying the foundation for advancing logical reasoning from qualitative analysis to quantitative research. Ying developed an approximate reasoning theory suitable for both propositional and predicate logics in the context of binary logic [2]. Subsequently, numerous scholars have conducted research on the grading of various logical systems based on truth degrees [3-5]. Reference [6] introduced a uniform probability measure into the binary propositional logic system, proposed the concept of truth degree for propositional logic formulas, defined similarity and pseudo-distance between formulas, constructed a logical metric space, and demonstrated the feasibility of conducting approximate reasoning research in this space. Following this, many scholars have extended these ideas and methods to various propositional logic systems and carried out extensive related research. Wang *et al*. established quantitative logic on multi-valued logic systems [7, 8], bridging the gap between mathematical logic and computational mathematics and providing a theoretical foundation for graded reasoning in logical systems. Building on this, references [9, 10] have conducted research on the divergence and compatibility of theories in quantitative logic. References [11, 12] have studied the axiomatic theory of truth degree and its operational properties in quantitative logic within predicate logic systems. References [13-17] have carried out related research on quantitative logic in different logical systems.

Although quantitative logic has propelled the degree-oriented development of logic, it faces two significant limitations: firstly, its lack of randomness renders it incapable of accurately capturing the stochastic nature of the truth or falsity of propositions in real-world scenarios; secondly, its failure to incorporate preconditions complicates the accurate reflection of the true degree of propositions under specific information constraints. To overcome these challenges, scholars have started integrating the concepts of randomization and conditional probability into propositional logic systems. References [18, 19] utilize a randomization method for assignment sets to propose the concept of randomized truth degree for formulas in propositional logic systems, establishing a randomized logical metric space and achieving the integration of probability logic and quantitative logic. Building on this, studies [20-22] investigate the randomized truth degree and randomized divergence of theories in many-valued propositional logic systems. References [23, 24] introduce the concepts of randomized truth degree and randomized divergence of theories for propositional formulas in fuzzy logic systems, and develop approximate reasoning, realizing the quantification of fuzzy logic systems. References [25, 26] introduce the concept of conditional truth degree based on conditional probability in binary and n-valued propositional logic systems, and study the corresponding logical inference rules. To adapt to the development of artificial intelligence, this degree-oriented approach to logical concepts based on randomness and conditional probability has become a hot topic of current research.

In order to characterize the randomized truth degree of a proposition in a fuzzy propositional logic system BL^*, this paper utilizes the concept of conditional probability and the randomization method of assignment sets to propose the conditional random truth degree of formulas. It proves the MP rule and HS rule for conditional randomized truth degrees. Simultaneously, the concepts of conditional randomized similarity and conditional randomized pseudo-distance between formulas are introduced, establishing a conditional randomized logical metric space. Within this conditional randomized logical metric space, the conditional randomized divergence of theories is proposed, and an initial study on approximate reasoning modes under given conditions is conducted. These research endeavors not only significantly enrich the theoretical landscape of fuzzy logic but also substantially enhance the capacity of related systems to address intricate challenges in domains such as uncertain information processing, fuzzy control, and approximate reasoning.

2. Preliminaries

Definition 2.1. ([5]) Let M be an algebra of type $(\neg, \vee, \rightarrow)$. Suppose there exists a partial order $\leq$ on M such that M becomes a bounded distributive lattice, where $\vee$ is the supremum operation with respect to the order $\leq$, and

$\neg$ is an order-reversing involution with respect to the order $\leq$. Furthermore, for every $a, b, c \in M$, the following conditions hold:

(M1) $\neg a \to \neg b = b \to a$,
(M2) $1 \to a = a, a \to a = 1$,
(M3) $b \to c \leq (a \to b) \to (a \to c)$,
(M4) $a \to (b \to c) = b \to (a \to c)$,
(M5) $a \to b \vee c = (a \to b) \vee (a \to c)$,
$a \to b \wedge c = (a \to b) \wedge (a \to c)$.

Here, 1 is the maximum element in M, and let $0 = \neg 1$. Then M is called a BR_0 −algebra.

Definition 2.2. ([5]) Let $S = \{q_1, q_2, \cdots\}$ be a set of atomic formulas, and $F(S)$ be the set of all formulas, i.e. $F(S)$ is the free algebra of type $(\neg, \vee, \to)$ generated by S. The formulas in $F(S)$ that take the following various forms are called axioms:

(BL^*1) $A \to (B \to A \wedge B)$,
(BL^*2) $(\neg A \to \neg B) \to (B \to A)$,
(BL^*3) $\big(A \to (B \to C)\big) \to \big(B \to (A \to C)\big)$,
(BL^*4) $(B \to C) \to \big((A \to B) \to (A \to C)\big)$,
(BL^*5) $A \to \neg\neg A$,
(BL^*6) $A \to A \vee B$,
(BL^*7) $A \vee B \to B \vee A$,
(BL^*8) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$,
(BL^*9) $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$.

The system composed of the formula set $F(S)$, axioms $1 \sim 9$, and the MP rule is called the BL^* system.

Definition 2.3. ([5]) In the BL^* system, the unary operation $\neg$ and binary operations $\vee, \to$ are defined on the interval [0,1] as follows:

$$\neg a = 1 - a,$$

$$a \vee b = max\{a, b\},$$

$$a \to b = \begin{cases} 1, a \leq b \\ \neg a \vee b, a > b \end{cases}.$$

Then [0,1] becomes a BR_0 −algebra, known as the BR_0 −unit interval. This corresponds to the operational definitions of the semantic theory for the BL^* fuzzy propositional logic system of type BR_0.

Definition 2.4. ([8]) Let $S = \{q_1, q_2, \cdots\}$ be a set of atomic formulas, and $F(S)$ be the set of all formulas, i.e., $F(S)$ is the $(\neg, \vee, \to)$type free algebra generated by S. In $F(S)$, let $A \wedge B$ represent $\neg(\neg A \vee \neg B)$ and $A \otimes B$ represent $\neg(A \to \neg B)$. $(\to, \otimes)$ is an adjoint pair satisfying $a \otimes b \leq c$ if and only if $a \leq b \to c, a, b, c \in [0,1]$.

Definition 2.5. ([23]) Let $A = A(q_1, q_2, \cdots, q_m) \in F(S)$ be a propositional formula containing m atomic formulas $q_1, q_2, \cdots, q_m$, with the assignment domain [0,1]. By replacing $q_1, q_2, \cdots, q_m$ in $A = A(q_1, q_2, \cdots, q_m)$ with numbers $x_1, x_2, \cdots, x_m$ from [0,1], and substituting the logical connectives $\neg, \vee, \to$ in A with the corresponding operations $\neg, \vee, \to$ in [0,1], we obtain an m-ary function $f_A(x_1, x_2, \cdots, x_m)$: $[0,1]^m \to [0,1]$. This function f_A is called the truth value function induced by the formula A. $\forall \hat{x} = (x_1, x_2, \cdots, x_m) \in [0,1]^m$, $f_A(\hat{x})$ is called an assignment of $A = A(q_1, q_2, \cdots, q_m)$.

Definition 2.6. ([23]) Let $p = \{p_1(x), p_2(x), \cdots, p_m(x), \cdots\}$ be a sequence of functions on [0,1], where $p_i(x)(i = 1,2, \cdots)$ is an assignment probability density function, meaning $p_i(x) \geq 0$ and satisfying $\int_0^1 p_i(x)\, dx = 1$, Such p is referred to as a sequence of assignment probability density functions, sometimes simply called probability density sequence. The set of all such assignment probability density function sequences is denoted by P.

Definition 2.7. ([23]) Let p be a sequence of valuation density functions, $\forall \hat{x} = (x_1, x_2, \cdots, x_m) \in [0,1]^m$, let $\varphi(\hat{x}) = p_1(x_1) \times p_2(x_2) \times \cdots \times p_m(x_m)$. Then we obtain a mapping $\varphi: [0,1]^m \to (0, +\infty)$, φ is called a randomized mapping density on $[0,1]^m$.

Definition 2.8. ([23]) Let p be a sequence of valuation density functions, $A = A(q_1, q_2, \cdots, q_m) \in F(S)$, and f_A be the truth function induced by the formula A. Define the m-fold integral as

$$\tau_p(A) = \int_{[0,1]^m} f_A(x_1, x_2, \cdots, x_m)\, p_1(x_1) p_2(x_2) \cdots p_m(x_m) dx_1 dx_2 \cdots dx_m.$$

Then the integral value $\tau_p(A)$ is called the $p-$randomized truth degree of the propositional formula A, or simply the randomized truth degree.

Proposition 2.1. ([23]) Let p be a sequence of valuation density functions, $\forall \hat{x} = (x_1, x_2, \cdots, x_m) \in [0,1]^m$, $\varphi(\hat{x}) > 0$, and for every assignment probability density function sequence on [0,1], it holds that $\int_0^1 \varphi(\hat{x})\, d\hat{x} = 1$. For convenience, let $\varphi(\hat{x}) = p_1(x_1) \times p_2(x_2) \times \cdots \times p_m(x_m)$,$d\hat{x} = dx_1 dx_2 \cdots dx_m$, then $\tau_p(A) = \int_{[0,1]^m} f_A(\hat{x})\, \varphi(\hat{x}) d\hat{x}$.

Proposition 2.2. ([23]) In the BL^* system, let $A, B \in F(S)$ and p is a sequence of valuation density functions, then:

(1) If $\vdash A \to B$, then$\tau_p(A) \le \tau_p(B)$,

(2) $\tau_p(A \vee B) = \tau_p(A) + \tau_p(B) - \tau_p(A \wedge B)$,

(3) If $A \approx B$, then $\tau_p(A) = \tau_p(B)$,

(4) $\tau_p(\neg A) = 1 - \tau_p(A)$.

Proposition 2.3. ([8]) In the BL^* system, let $\Gamma \subset F(S), A, B \in F(S)$, then the following generalized deduction theorem holds:

$$\Gamma \cup \{A\} \vdash B \text{ if and only if } \Gamma \vdash A^2 \to B.$$

3. Conditional Randomized Truth Degree in Fuzzy Logic

Traditional truth degree theories suffer from two main limitations: firstly, they lack randomness and thus cannot accommodate the probabilistic nature of proposition assignments in real-world scenarios; secondly, they fail to consider the constraints imposed by preconditions, limiting their ability to depict only the absolute truth degrees of propositions without any additional conditions. However, logical reasoning in reality is almost always based on specific premises, rendering the absolute truth degrees without preconditions of limited practical significance for guiding actual reasoning processes. Therefore, the introduction of conditional randomized truth degrees enables the construction of a truth degree measurement system that incorporates both randomness and conditional constraints. This approach addresses the issue where traditional truth degrees and ordinary randomized truth degrees are unable to characterize "fuzzy random logical reasoning with premises", thereby enhancing the research on the gradation and probabilization of fuzzy propositional logic systems.

Building upon the semantics of the BR_0-algebra within the fuzzy propositional logic system BL^*, which operates over the assignment domain [0,1], we introduce a "squared conjunction" operation for preconditions. By integrating the theory of randomized truth degrees with the concept of conditional probability, we propose the notion of the conditional randomized truth degree. This concept accurately quantifies the truth degree of a formula A under the constraint of a specific precondition Σ, thereby addressing the limitations inherent in traditional truth degree theories that neglect to account for preconditions.

Definition 3.1. In the BL^* system, let $A \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$,$\otimes \Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Define

$$\tau_p(A|\Sigma)=\frac{\tau_p\big(A\otimes(\otimes\Sigma^2)\big)}{\tau_p(\otimes\Sigma^2)}.$$

Then $\tau_p(A|\Sigma)$ is called the conditional randomized truth degree of formula A under the condition Σ, or simply the Σ −conditional randomized truth degree. When Σ consists entirely of theorems, $\otimes\Sigma^2$ is a tautology, so $\tau_p(A|\Sigma)=\tau_p(A)$; when Σ contains only one formula B, then $\tau_p(A|B)=\frac{\tau_p(A\otimes B^2)}{\tau_p(B^2)}$.

Theorem 3.1. In the BL^* system, let $A,B\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$,$\otimes\Sigma^2={A_1}^2\otimes{A_2}^2\otimes\cdots{A_n}^2$, $\tau_p(\otimes\Sigma^2)>0$, and p is a sequence of valuation density functions. Then

(1) $0\le\tau_p(A|\Sigma)\le 1$;

(2) $\tau_p(A\vee B|\Sigma)=\tau_p(A|\Sigma)+\tau_p(B|\Sigma)-\tau_p(A\wedge B|\Sigma)$;

(3) If $A\otimes(\otimes\Sigma^2)$ and $B\otimes(\otimes\Sigma^2)$ are logically incompatible, then$\tau_p(A\vee B|\Sigma)=\tau_p(A|\Sigma)+\tau_p(B|\Sigma)$.

Proof. (1) Since $\vdash A\otimes(\otimes\Sigma^2)\to(\otimes\Sigma^2)$, it follows from Proposition 2.2 that $\tau_p\big(A\otimes(\otimes\Sigma^2)\big)\le\tau_p(\otimes\Sigma^2)$. Furthermore, since $\tau_p\big(A\otimes(\otimes\Sigma^2)\big)\ge 0$ and according to Definition 3.1, $\tau_p(A|\Sigma)\le 1$, therefore $0\le\tau_p(A|\Sigma)\le 1$;

(2) Because

$$\tau_p\big((A\vee B)\otimes(\otimes\Sigma^2)\big)=\tau_p\Big(\big(A\otimes(\otimes\Sigma^2)\big)\vee\big(B\otimes(\otimes\Sigma^2)\big)\Big)$$

$$=\tau_p\big(A\otimes(\otimes\Sigma^2)\big)+\tau_p\big(B\otimes(\otimes\Sigma^2)\big)-\tau_p\Big(\big(A\otimes(\otimes\Sigma^2)\big)\wedge\big(B\otimes(\otimes\Sigma^2)\big)\Big),$$

dividing both sides by $\tau_p(\otimes\Sigma^2)$ simultaneously, we get $\tau_p(A\vee B|\Sigma)=\tau_p(A|\Sigma)+\tau_p(B|\Sigma)-\tau_p(A\wedge B|\Sigma)$;

(3) If $A\otimes(\otimes\Sigma^2)$ and $B\otimes(\otimes\Sigma^2)$ are logically incompatible, then $\big(A\otimes(\otimes\Sigma^2)\big)\wedge\big(B\otimes(\otimes\Sigma^2)\big)$ is a contradiction, and thus $\tau_p\Big(\big(A\otimes(\otimes\Sigma^2)\big)\wedge\big(B\otimes(\otimes\Sigma^2)\big)\Big)=0$. Therefore, it follows from (2) that

$$\tau_p\big((A\vee B)\otimes(\otimes\Sigma^2)\big)=\tau_p\big(A\otimes(\otimes\Sigma^2)\big)+\tau_p\big(B\otimes(\otimes\Sigma^2)\big).$$

Dividing both sides by $\tau_p(\otimes\Sigma^2)$ simultaneously, we get $\tau_p(A\vee B|\Sigma)=\tau_p(A|\Sigma)+\tau_p(B|\Sigma)$.

Theorem 3.2. In the BL^* system, let $A,B\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$,$\otimes\Sigma^2={A_1}^2\otimes{A_2}^2\otimes\cdots{A_n}^2$, $\tau_p(\otimes\Sigma^2)>0$, and p is a sequence of valuation density functions. Then

(1) If $\Sigma\vdash A$, then $\tau_p(A|\Sigma)=1$;

(2) If $\Sigma\cup\{B\}\vdash A$, then $\tau_p(B^2|\Sigma)\le\tau_p(A|\Sigma)$;

(3) If $\Sigma\cup\{B\}\vdash A^2$, and $\Sigma\cup\{A\}\vdash B^2$, then $\tau_p(A^2|\Sigma)=\tau_p(B^2|\Sigma)$.

Proof. (1) If $\Sigma\vdash A$, then $\vdash(\otimes\Sigma^2)\to A$, so $\vdash(\otimes\Sigma^2)\to\big(A\otimes(\otimes\Sigma^2)\big)$, and thus $\tau_p(\otimes\Sigma^2)\le\tau_p\big(A\otimes(\otimes\Sigma^2)\big)$.

Therefore,

$$\tau_p(A|\Sigma)=\frac{\tau_p(A\otimes\Sigma^2)}{\tau_p(\Sigma^2)}\ge 1.$$

Since

$$\tau_p(A|\Sigma)=\frac{\tau_p(A\otimes\Sigma^2)}{\tau_p(\Sigma^2)}\le 1,$$

it follows that $\tau_p(A|\Sigma) = 1$;

(2) If $\Sigma \cup \{B\} \nvdash A$, then according to Proposition 2.3, we have $\nvdash \left(B^2 \otimes (\otimes\, \Sigma^2)\right) \to A$, which implies

$$\nvdash \left(B^2 \otimes (\otimes\, \Sigma^2)\right) \to \left(A \otimes (\otimes\, \Sigma^2)\right).$$

Therefore, by Proposition 2.2, it follows that

$$\tau_p\left(B^2 \otimes (\otimes\, \Sigma^2)\right) \le \tau_p\left(A \otimes (\otimes\, \Sigma^2)\right),$$

so $\tau_p(B^2|\Sigma) \le \tau_p(A|\Sigma)$;

(3) If $\Sigma \cup \{B\} \nvdash A^2$, then it follows from (3) that $\tau_p(B^2|\Sigma) \le \tau_p(A^2|\Sigma)$. If $\Sigma \cup \{A\} \nvdash B^2$, then it follows from (3) that $\tau_p(A^2|\Sigma) \le \tau_p(B^2|\Sigma)$. Therefore, If $\Sigma \cup \{B\} \nvdash A^2$, and $\Sigma \cup \{A\} \nvdash B^2$, then $\tau_p(A^2|\Sigma) = \tau_p(B^2|\Sigma)$.

Theorem 3.3. In the BL^* system, let $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$,$\otimes\, \Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes\, \Sigma^2) > 0$, and p is a sequence of valuation density functions. Then

(1) If $A \approx B$, then $\tau_p(A|\Sigma) = \tau_p(B|\Sigma)$;

(2) $\tau_p(\neg A|\Sigma) = 1 - \tau_p(A|\Sigma)$;

(3) If $\tau_p(A|\Sigma) \ge \alpha, \tau_p(A \to B|\Sigma) \ge \beta$, then $\tau_p(B|\Sigma) \ge \alpha + \beta - 1$;

(4) If $\tau_p(A \to B|\Sigma) \ge \alpha, \tau_p(B \to C|\Sigma) \ge \beta$, then $\tau_p(A \to C|\Sigma) \ge \alpha + \beta - 1$;

(5) $\tau_p(A \to B|\Sigma) = \tau_p(A \wedge B|\Sigma) - \tau_p(A|\Sigma) + 1$.

Proof. (1) If $A \approx B$, then $(A \to B) \wedge (B \to A)$ is a tautology. Therefore, any assignment that makes $\otimes\, \Sigma$ true also makes A true if and only if it makes B true. Hence, $\tau_p(A|\Sigma) = \tau_p(B|\Sigma)$;

(2) Because

$$\tau_p(\otimes\, \Gamma^2) = \tau_p\left((A \vee \neg A) \otimes (\otimes\, \Sigma^2)\right) = \tau_p\left(A \otimes (\otimes\, \Sigma^2)\right) + \tau_p\left(\neg A \otimes (\otimes\, \Sigma^2)\right) - \tau_p\left((A \wedge \neg A) \otimes (\otimes\, \Sigma^2)\right),$$

and

$$\tau_p\left((A \wedge \neg A) \otimes (\otimes\, \Sigma^2)\right) = 0,$$

it follows that

$$\tau_p(\otimes\, \Sigma) = \tau_p\left(A \otimes (\otimes\, \Sigma^2)\right) + \tau_p\left(\neg A \otimes (\otimes\, \Sigma^2)\right).$$

Therefore, dividing both sides by $\tau_p(\otimes\, \Sigma^2)$ simultaneously, we get

$$1 = \tau_p(A|\Sigma) + \tau_p(\neg A|\Sigma),$$

which implies $\tau_p(\neg A|\Sigma) = 1 - \tau_p(A|\Sigma)$;

(3) Since $\tau_p(A|\Sigma) \ge \alpha, \tau_p(A \to B|\Sigma) \ge \beta$, according to Definition 3.1, we have $\tau_p\left(A \otimes (\otimes\, \Sigma^2)\right) \ge \alpha \tau_p(\otimes\, \Sigma^2)$, $\tau_p\left((A \to B) \otimes (\otimes\, \Sigma^2)\right) \ge \beta \tau_p(\otimes\, \Sigma^2)$. By Proposition 2.1, it follows that

$$\tau_p(\otimes\,\Sigma^2)+\tau_p\big(B\otimes(\otimes\,\Sigma^2)\big)=\int_{[0,1]^m} f_{\otimes\Sigma^2}(\chi)\,\varphi(\chi)d\chi+\int_{[0,1]^m} f_{B\otimes(\otimes\Sigma^2)}(\chi)\,\varphi(\chi)d\chi$$

$$=\int_{[0,1]^m}\Big(f_{\otimes\Sigma^2}(\chi)+f_{B\otimes(\otimes\Sigma^2)}(\chi)\Big)\varphi(\chi)d\chi$$

$$\geq\int_{[0,1]^m}\Big(f_{A\otimes(\otimes\Sigma^2)}(\chi)+f_{(A\to B)\otimes(\otimes\Sigma^2)}(\chi)\Big)\varphi(\chi)d\chi$$

$$=\int_{[0,1]^m} f_{A\otimes(\otimes\Sigma^2)}(\chi)\,\varphi(\chi)d\chi+\int_{[0,1]^m} f_{(A\to B)\otimes(\otimes\Sigma^2)}(\chi)\,\varphi(\chi)d\chi$$

$$=\tau_p\big(A\otimes(\otimes\,\Sigma^2)\big)+\tau_p\big((A\to B)\otimes(\otimes\,\Sigma^2)\big)$$

$$\geq\alpha\tau_p(\otimes\,\Sigma^2)+\beta\tau_p(\otimes\,\Sigma^2)$$

$$=(\alpha+\beta)\tau_p(\otimes\,\Sigma^2),$$

Dividing both sides by $\tau_p(\otimes\,\Sigma^2)$ simultaneously, we obtain $\tau_p(B|\Sigma)\geq\alpha+\beta-1$;

(4) Since $\Sigma\not\vdash(B\to C)\to\big((A\to B)\to(A\to C)\big)$, it follows from Theorem 3.2 (1) that

$$\tau_p\Big((B\to C)\to\big((A\to B)\to(A\to C)\big)|\Sigma\Big)=1.$$

Given that $\tau_p(B\to C|\Sigma)\geq\beta$, it can be inferred from (3) that

$$\tau_p\Big(\big((A\to B)\to(A\to C)\big)|\Sigma\Big)\geq 1+\beta-1=\beta.$$

Additionally, since $\tau_p(A\to B|\Sigma)\geq\alpha$, it can be deduced from (3) that $\tau_p(A\to C|\Sigma)\geq\alpha+\beta-1$;

(5) It follows from Theorem 3.1 (2) and Theorem 3.3 (2) that

$$\tau_p(A\to B|\Sigma)=\tau_p(\neg A\vee B|\Sigma)=\tau_p(\neg A|\Sigma)+\tau_p(B|\Sigma)-\tau_p(\neg A\wedge B|\Sigma)$$

$$=1-\tau_p(A|\Sigma)+\tau_p(A\wedge B|\Sigma)=\tau_p(A\wedge B|\Sigma)-\tau_p(A|\Sigma)+1.$$

Example 3.1. In the BL^* system, let $A=q_1, B=q_2, \Gamma=\{A,B\}$, and consider two sequences of probability density functions: $p_1(x_1)=3{x_1}^2, p_2(x_2)=3(1-x_2)^2$, Calculate $\tau_p(B|A)$ and $\tau_p(A|B)$.

Solution. Since

$$\tau_p(A^2)=\tau_p({q_1}^2)=\int_0^1(x\otimes x)p_1(x)dx=\int_{\frac{1}{2}}^1(2x_1-1)\cdot 3{x_1}^2dx_1=\frac{7}{16},$$

$$\tau_p(B^2)=\tau_p({q_2}^2)=\int_0^1(x\otimes x)p_2(x)dx=\int_{\frac{1}{2}}^1(2x_2-1)\cdot 3(1-x_2)^2dx_2=\frac{1}{32},$$

$$\tau_p(B\otimes A^2)=\tau_p(q_2\otimes {q_1}^2)=\int_{\frac{1}{2}}^1\int_{2-2x_1}^1(x_2+2x_1-2)\cdot 3{x_1}^2\cdot 3(1-x_2)^2dx_2dx_1=\frac{49}{1536},$$

$$\tau_p(A\otimes B^2)=\tau_p(q_1\otimes {q_2}^2)=\int_{\frac{1}{2}}^1\int_{2-2x_2}^1(x_1+2x_2-2)\cdot 3{x_1}^2\cdot 3(1-x_2)^2dx_1dx_2=\frac{11}{1280},$$

it follows from Definition 3.1 that

$$\tau_p(B|A)=\frac{\tau_p(B\otimes A^2)}{\tau_p(A^2)}=\frac{7}{96},\text{ and }\tau_p(A|B)=\frac{\tau_p(A\otimes B^2)}{\tau_p(B^2)}=\frac{11}{40}.$$

Next, based on the relevant conclusions of conditional random truth degree, we will establish a conditional random logical metric space.

4. Conditional Randomized Logical Metric Space

With the concept and associated conclusions of the conditional randomized truth degree firmly established, we are now in a position to delve into an examination of the conditional randomized similarity among propositional formulas. This inquiry naturally leads us to introduce the concept of conditional randomized pseudo-distance between these formulas. By leveraging this notion of conditional randomized pseudo-distance, we can construct a conditional randomized logical metric space. This novel metric space addresses a significant limitation inherent in traditional randomized logical metric spaces, namely their inability to incorporate precondition constraints. It empowers the logical space to accurately characterize metric relationships between propositions under specific conditional constraints, thereby substantially enriching the research landscape concerning the gradation of fuzzy logic systems.

Definition 4.1. In the BL^* system, let $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\otimes \Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Define

$$\xi_\Sigma(A, B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big).$$

Then $\xi_\Sigma(A, B|\Sigma)$ is called the conditional randomized similarity degree between A and B under condition Σ, or simply the $\Sigma -$conditional randomized similarity degree. According to Definition 3.1, we have $\xi_\Sigma(A, B|\Sigma) = \frac{\tau_p\left((A\to B)\wedge(B\to A)\otimes(\otimes\Sigma^2)\right)}{\tau_p(\otimes\Sigma^2)}$. Clearly, $\xi_\Sigma(A, B|\Sigma) = \xi_\Sigma(B, A|\Sigma)$.

Theorem 4.1. In the BL^* system, let $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Then

$$\xi_\Sigma(A, B|\Sigma) = \tau_p(A \to B|\Sigma) + \tau_p(B \to A|\Sigma) - 1 = 1 - \tau_p\big((A \vee B)|\Sigma\big) + \tau_p\big((A \wedge B)|\Sigma\big).$$

Proof. From Definition 4.1 and Theorem 3.1 (2), it follows that

$$\xi_\Sigma(A, B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = \tau_p(A \to B|\Sigma) + \tau_p(B \to A|\Sigma) - \tau_p\big((A \to B) \vee (B \to A)|\Sigma\big).$$

Since $(A \to B) \vee (B \to A)$ is a tautology, by Theorem 3.2 (1) we have $\tau_p\big((A \to B) \vee (B \to A)|\Sigma\big) = 1$, hence

$$\xi_\Sigma(A, B|\Sigma) = \tau_p(A \to B|\Sigma) + \tau_p(B \to A|\Sigma) - 1.$$

Combining this conclusion with Theorem 3.3 (5), we obtain

$$\begin{aligned}
&\xi_\Sigma(A, B|\Sigma) = \tau_p(A \to B|\Sigma) + \tau_p(B \to A|\Sigma) - 1 \\
&= \tau_p(A \wedge B|\Sigma) - \tau_p(A|\Sigma) + 1 + \tau_p(B \wedge A|\Sigma) - \tau_p(B|\Sigma) + 1 - 1 \\
&= 2\tau_p(A \wedge B|\Sigma) - \tau_p(A|\Sigma) - \tau_p(B|\Sigma) + 1 \\
&= \tau_p(A \wedge B|\Sigma) + \tau_p(A|\Sigma) + \tau_p(B|\Sigma) - \tau_p(A \vee B|\Sigma) - \tau_p(A|\Sigma) - \tau_p(B|\Sigma) + 1 \\
&= 1 - \tau_p(A \vee B|\Sigma) + \tau_p(A \wedge B|\Sigma).
\end{aligned}$$

Therefore, it can be concluded that

$$\xi_\Sigma(A, B|\Sigma) = \tau_p(A \to B|\Sigma) + \tau_p(B \to A|\Sigma) - 1 = 1 - \tau_p\big((A \vee B)|\Sigma\big) + \tau_p\big((A \wedge B)|\Sigma\big).$$

Theorem 4.2. In the BL^* system, let $A, B, C \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Then

$$\xi_\Sigma(A, B|\Sigma) + \xi_\Sigma(B, C|\Sigma) \leq 1 + \xi_\Sigma(A, C|\Sigma).$$

Proof. If $A, B, C \in F(S)$, then by Theorem 4.1 it follows that

$$\xi_\Sigma(A,B|\Sigma)=\tau_p(A\to B|\Sigma)+\tau_p(B\to A|\Sigma)-1,$$

and thus

$$\xi_\Sigma(A,B|\Sigma)+\xi_\Sigma(B,C|\Sigma)=\tau_p(A\to B|\Sigma)+\tau_p(B\to A|\Sigma)-1+\tau_p(B\to C|\Sigma)+\tau_p(C\to B|\Sigma)-1$$

$$=\left[\tau_p(A\to B|\Sigma)+\tau_p(B\to C|\Sigma)-1\right]+\left[\tau_p(C\to B|\Sigma)+\tau_p(B\to A|\Sigma)-1\right].$$

By Theorem 3.3(4), we have

$$\tau_p(A\to B|\Sigma)+\tau_p(B\to C|\Sigma)-1\le\tau_p(A\to C|\Sigma),\tau_p(C\to B|\Sigma)+\tau_p(B\to A|\Sigma)-1\le\tau_p(C\to A|\Sigma),$$

so combining this with Theorem 4.1, it can be concluded that

$$\xi_\Sigma(A,B|\Sigma)+\xi_\Sigma(B,C|\Sigma)=\left[\tau_p(A\to B|\Sigma)+\tau_p(B\to C|\Sigma)-1\right]+\left[\tau_p(C\to B|\Sigma)+\tau_p(B\to A|\Sigma)-1\right]$$

$$\le\tau_p(A\to C|\Sigma)+\tau_p(C\to A|\Sigma)=\xi_\Sigma(A,C|\Sigma)+1,$$

which means $\xi_\Sigma(A,B|\Sigma)+\xi_\Sigma(B,C|\Sigma)\le 1+\xi_\Sigma(A,C|\Sigma)$.

Definition 4.2. In the BL^* system, let $A,B\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$, $\tau_p(\otimes\Sigma^2)>0$, and p is a sequence of valuation density functions. Define

$$d_\Sigma(A,B|\Sigma)=1-\xi_\Sigma(A,B|\Sigma).$$

Then $d_\Sigma(A,B|\Sigma)$ is called the conditional randomized pseudo-metric between A and B under condition Σ, or simply the Σ −conditional randomized pseudo-metric.

Proposition 4.1. In the BL^* system, let $A,B,C\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$, $\tau_p(\otimes\Sigma^2)>0$, and p is a sequence of valuation density functions. If $d_\Sigma(A,B|\Sigma)=1-\xi_\Sigma(A,B|\Sigma)$ represents the conditional randomized pseudo-metric between formulas$A$ and B under condition Σ, then

(1) $d_\Sigma(A,A|\Sigma)=0$;
(2) $d_\Sigma(A,B|\Sigma)=d_\Sigma(B,A|\Sigma)$;
(3) $d_\Sigma(A,C|\Sigma)\le d_\Sigma(A,B|\Sigma)+d_\Sigma(B,C|\Sigma)$.

Proof. (1) From Definition 4.1, it follows that

$$\xi_\Sigma(A,A|\Sigma)=\tau_p\big((A\to A)\wedge(A\to A)|\Sigma\big)=1.$$

Therefore, according to Definition 4.2, we have

$$d_\Sigma(A,A|\Sigma)=1-\xi_\Sigma(A,A|\Sigma)=0.$$

(2) From Definition 4.1, it follows that

$$\xi_\Sigma(A,B|\Sigma)=\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big)=\tau_p\big((B\to A)\wedge(A\to B)|\Sigma\big)=\xi_\Sigma(B,A|\Sigma).$$

Therefore, according to Definition 4.2, we have

$$d_\Sigma(A,B|\Sigma)=1-\xi_\Sigma(A,B|\Sigma)=1-\xi_\Sigma(B,A|\Sigma)=d_\Sigma(B,A|\Sigma).$$

(3) From Definition 4.2, it follows that

$$d_\Sigma(A,C|\Sigma)=1-\xi_\Sigma(A,C|\Sigma).$$

By Theorem 4.3, we have

$$\xi_\Sigma(A,C|\Sigma)\ge\xi_\Sigma(A,B|\Sigma)+\xi_\Sigma(B,C|\Sigma)-1.$$

Therefore,

$$d_{\Sigma}(A, C|\Sigma) = 1 - \xi_{\Sigma}(A, C|\Sigma) \leq 1 - [\xi_{\Sigma}(A, B|\Sigma) + \xi_{\Sigma}(B, C|\Sigma) - 1] = d_{\Sigma}(A, B|\Sigma) + d_{\Sigma}(B, C|\Sigma).$$

Theorem 4.3. In the BL^* system, let $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Then

(1) $A \approx B$, if and only if $d_{\Sigma}(A, B|\Sigma) = 0$;

(2) $A \approx \neg B$, if and only if $d_{\Sigma}(A, B|\Sigma) = 1$;

Proof. (1) If $A \approx B$, then $A \to B$ and $B \to A$ are tautologies, which implies that $(A \to B) \wedge (B \to A)$ is also a tautology. According to Theorem 3.2 (1), we have $\tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = 1$, meaning

$$\xi_{\Sigma}(A, B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = 1.$$

Conversely, if $\xi_{\Sigma}(A, B|\Sigma) = 1$, then $\tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = 1$. By Theorem 3.2(1), $(A \to B) \wedge (B \to A)$ is a tautology, implying $A \approx B$. Therefore, $A \approx B$ if and only if $\xi_{\Sigma}(A, B|\Sigma) = 1$. According to Definition 4.2, $d_{\Sigma}(A, B|\Sigma) = 1 - \xi_{\Sigma}(A, B|\Sigma)$, so $A \approx B$, if and only if $d_{\Sigma}(A, B|\Sigma) = 0$;

(2) If $A \approx \neg B$, then $A \to \neg B$ and $\neg B \to A$ are tautologies, which implies that $(A \to \neg B) \wedge (\neg B \to A)$ is a tautology, and thus $\neg\big((A \to B) \wedge (B \to A)\big)$ is also a tautology. By Theorem 3.2(1), we have

$$\tau_p\big((A \to \neg B) \wedge (\neg B \to A)|\Sigma\big) = \tau_p\big(\neg\big((A \to B) \wedge (B \to A)\big)|\Sigma\big) = 1.$$

According to Theorem 3.3 (2), it follows that $\tau_p(\neg A|\Sigma) = 1 - \tau_p(A|\Sigma)$, so

$$\tau_p\big(\neg\big((A \to B) \wedge (B \to A)\big)|\Sigma\big) = 1 - \tau_p\Big(\big((A \to B) \wedge (B \to A)\big)|\Sigma\Big) = 1.$$

Therefore, $\tau_p\Big(\big((A \to B) \wedge (B \to A)\big)|\Sigma\Big) = 0$, which means $\xi_{\Sigma}(A, B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = 0$.

Conversely, if $\xi_{\Sigma}(A, B|\Sigma) = 0$, then $\tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) = 0$. By Theorem 3.3 (2), we can infer that

$$\tau_p\big(\neg\big((A \to B) \wedge (B \to A)\big)|\Sigma\big) = 1 - \tau_p\Big(\big((A \to B) \wedge (B \to A)\big)|\Sigma\Big) = 1.$$

According to Theorem 3.2(1), $\neg\big((A \to B) \wedge (B \to A)\big)$ is a tautology, which implies that $(A \to \neg B) \wedge (\neg B \to A)$ is also a tautology, so $A \approx \neg B$. In conclusion, $A \approx \neg B$ if and only if $\xi_{\Sigma}(A, B|\Sigma) = 0$. According to Definition 4.2, $d_{\Sigma}(A, B|\Sigma) = 1 - \xi_{\Sigma}(A, B|\Sigma)$, so $A \approx \neg B$, if and only if $d_{\Sigma}(A, B|\Sigma) = 1$.

Definition 4.3. In the BL^* system, let p be a sequence of valuation density functions, $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$. If d_{Σ} is the conditional randomized pseudo-metric between formulas in $F(S)$, then $(F(S), d_{\Sigma})$ is called a conditional randomized logical metric space.

Proposition 4.2. In the BL^* system, let $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\tau_p(\otimes \Sigma^2) > 0$, and p is a sequence of valuation density functions. Then in the conditional randomized logical metric space $(F(S), d_{\Sigma})$, it has the following properties:

(1) $0 \leq d_{\Sigma}(A, B|\Sigma) \leq 1$;

(2) $d_{\Sigma}(A, B|\Sigma) = \tau_p(A \vee B|\Sigma) - \tau_p(A \wedge B|\Sigma)$.

Proof. (1) By Theorem 3.1, we know that $0 \leq \tau_p(A|\Sigma) \leq 1$. According to Definition 4.1,

$$\xi_{\Sigma_p}(A, B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big),$$

so $0 \leq \xi_{\Sigma_p}(A, B|\Sigma) \leq 1$.

From Definition 4.1, it follows that $d_{\Sigma_p}(A,B|\Sigma) = 1 - \xi_{\Sigma_p}(A,B|\Sigma)$, therefore $0 \le d_{\Sigma_p}(A,B|\Sigma) \le 1$;

(2) According to Definition 4.2, $d_\Sigma(A,B|\Sigma) = 1 - \xi_{\Sigma_p}(A,B|\Sigma)$. From Definition 4.1, we know that

$$\xi_\Sigma(A,B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big).$$

Therefore,

$$\begin{aligned} d_\Sigma(A,B|\Sigma) &= 1 - \xi_\Sigma(A,B|\Sigma) = 1 - \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) \\ &= 1 - \tau_p(A \leftrightarrow B|\Sigma) = 1 - \tau_p\big((\neg A \wedge \neg B) \vee (A \wedge B)|\Sigma\big) \\ &= 1 - \Big(\tau_p(\neg A \wedge \neg B|\Sigma) + \tau_p(A \wedge B|\Sigma)\Big) = 1 - \Big(\tau_p(\neg(A \vee B)|\Sigma) + \tau_p(A \wedge B|\Sigma)\Big) \\ &= 1 - \Big(1 - \tau_p(A \vee B|\Sigma) + \tau_p(A \wedge B|\Sigma)\Big) = \tau_p(A \vee B|\Sigma) - \tau_p(A \wedge B|\Sigma) \end{aligned}$$

that is, $d_\Sigma(A,B|\Sigma) = \tau_p(A \vee B|\Sigma) - \tau_p(A \wedge B|\Sigma)$.

5. Conditional Randomized Divergence of a Theory and Approximate Reasoning

A subset Γ of the formula set $F(S)$ is called a theory. A theory Γ is said to be consistent if no contradiction can be deduced from it; otherwise, it is inconsistent. For an inconsistent theory Γ, its set of all consequences $D(\Gamma)$ is equal to $F(S)$, thus the structure of $D(\Gamma)$ for an inconsistent theory Γ is well-defined. However, the structure of $D(\Gamma)$ can vary significantly for consistent theories Γ, which renders it of great research value to further classify the consistency degrees of consistent theories. In the research framework of classical quantitative logic, the consistency degree of a theory can be characterized by the divergence degree of the theory. We first introduce the concept of conditional randomized divergence degree within the framework of a conditional randomized logical metric space in the fuzzy propositional logic systemBL^*.

Definition 5.1. In the BL^* system, let p be a sequence of valuation density functions, $A,B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\otimes\Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes\Sigma^2) > 0$, Γ is a theory in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ. Define

$$div_\Sigma(\Gamma|\Sigma) = sup\{d_\Sigma(A,B|\Sigma)|A,B \in D(\Gamma)\}.$$

Then $div_\Sigma(\Gamma|\Sigma)$ is called the $\Sigma-$ conditional randomized divergency degree of Γ. When $div_\Sigma(\Gamma|\Sigma) = 1$, the theory Γ is called Σ $-$conditionally randomly fully divergent.

Theorem 5.1. In the BL^* system, let p be a sequence of valuation density functions, $A,B,C \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\otimes\Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes\Sigma^2) > 0$, Γ is a theory in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ. Then

$$div_\Sigma(\Gamma|\Sigma) = 1 - inf\{\tau_p(C|\Sigma)|C \in D(\Gamma)\}.$$

Proof. From Definition 4.1 and Definition 4.2, we have

$$\xi_\Sigma(A,B|\Sigma) = \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) \text{ and } d_\Sigma(A,B|\Sigma) = 1 - \xi_\Sigma(A,B|\Sigma).$$

Therefore, $d_\Sigma(A,B|\Sigma) = 1 - \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big)$.

Let $A,B \in D(\Gamma)$, then $A \wedge B \in D(\Gamma)$. From Definition 2.8 and Definition 3.1, we obtain:

When $f_A(a) \le f_B(a)$, we have $f_{(A\to B)\wedge(B\to A)}(a) = f_{(\neg B \vee A)}(a)$. Since $f_{(\neg B\vee A)}(a) \ge f_{A\wedge B}(a)$, it follows that $f_{(A\to B)\wedge(B\to A)}(a) \ge f_{A\wedge B}(a)$. Therefore, $1 - \tau_p\big((A \to B) \wedge (B \to A)|\Sigma\big) \le 1 - \tau_p(A \wedge B|\Sigma)$.

When $f_A(a) > f_B(a)$, we have $f_{(A\to B)\wedge(B\to A)}(a) = f_{(\neg A\vee B)}(a)$. Since $f_{(\neg A\vee B)}(a) \geq f_{A\wedge B}(a)$, it follows that $f_{(A\to B)\wedge(B\to A)}(a) \geq f_{A\wedge B}(a)$. Therefore, $1-\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big) \leq 1-\tau_p(A\wedge B|\Sigma)$.

In summary, $1-\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big) \leq 1-\tau_p(A\wedge B|\Sigma)$.

Thus,

$$div_\Sigma(\Gamma|\Sigma) = sup\{d_\Sigma(A,B|\Sigma)|A,B\in D(\Gamma)\} = sup\{1-\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big)|A,B\in D(\Gamma)\}$$

$$= 1-inf\{\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big)|A,B\in D(\Gamma)\},$$

$$\leq 1-inf\{\tau_p(A\wedge B|\Sigma)|A,B\in D(\Gamma)\}.$$

Similarly, it can be proven that $1-\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big) \geq 1-\tau_p(A\vee B|\Sigma)$, so

$$div_\Sigma(\Gamma|\Sigma) \geq 1-inf\{\tau_p(A\vee B|\Sigma)|A,B\in D(\Gamma)\}.$$

Therefore, we conclude that $div_\Sigma(\Gamma|\Sigma) = 1-inf\{\tau_p(A\vee B|\Sigma)|A,B\in D(\Gamma)\}$.

Given the arbitrariness of $A,B\in D(\Gamma)$, it follows that $div_\Sigma(\Gamma|\Sigma) = 1-inf\{\tau_p(C|\Sigma)|C\in D(\Gamma)\}$.

Theorem 5.2. In the BL^* system, let p be a sequence of valuation density functions, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$, $\otimes\Sigma^2 = {A_1}^2\otimes {A_2}^2\otimes\cdots {A_n}^2$, $\tau_p(\otimes\Sigma^2)>0$, $\Gamma=\{q_1,q_2,\cdots,q_n\}$ is a theory in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ. Then

$$div_\Sigma(\Gamma|\Sigma) = 1-\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma).$$

Proof. If $\Gamma=\{q_1,q_2,\cdots,q_n\}$ is a finite theory, then ${q_1}^2\otimes{q_2}^2\otimes\cdots{q_n}^2\in D(\Gamma)$ is the formula with the minimum $\Sigma-$conditional randomized truth degree in $D(\Gamma)$, and thus we have

$$inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\} = \tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma).$$

From Theorem 5.1, it follows that

$$div_\Sigma(\Gamma|\Sigma) = 1-inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\} = 1-\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma).$$

If $\Gamma=\{q_1,q_2,\cdots,q_n,\cdots\}$ is an infinite theory, then for any $n\in N$, we have $q_1\wedge q_2\wedge\cdots\wedge q_n\in D(\Gamma)$. Since $\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma)$ is monotonically decreasing, it follows that

$$inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\} \geq \lim_{n\to\infty}\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma).$$

Furthermore, because $\lim_{n\to\infty}\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma) \geq inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\}$ also holds,

we conclude that $inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\} = \lim_{n\to\infty}\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma)$.

From Theorem 5.1, it follows that

$$div_\Sigma(\Gamma|\Sigma) = 1-inf\{\tau_p(A|\Sigma)|A\in D(\Gamma)\} = 1-\tau_p(q_1\wedge q_2\wedge\cdots\wedge q_n|\Sigma).$$

Example 5.1. In the BL^* system, let$A=q_1, B=q_2, \Gamma=\{A,B\}, \Sigma=\{A\}$, and consider two sequences of probability density functions:$p_1(x_1)=2x_1, p_2(x_2)=2-2x_2$, Calculate $\tau_p(B|A), \tau_p(A|B), \tau_p(A\wedge B|A)$, and $div_\Sigma(\Gamma|\Sigma)$.

Solution. Since

$$\tau_p(A^2) = \tau_p({q_1}^2) = \int_0^1 (x \otimes x) p_1(x) dx = \int_{\frac{1}{2}}^1 x \cdot 2x dx = \frac{7}{12},$$

$$\tau_p(B^2) = \tau_p({q_2}^2) = \int_0^1 (x \otimes x) p_2(x) dx = \int_{\frac{1}{2}}^1 x \cdot (2-2x) dx = \frac{1}{6},$$

$$\tau_p(B \otimes A^2) = \tau_p(q_2 \otimes {q_1}^2) = \frac{8}{3}\int_0^{\frac{1}{2}} (1-x) \cdot x^3 dx = \frac{49}{240},$$

$$\tau_p(A \otimes B^2) = \tau_p(q_1 \otimes {q_2}^2) = 4\int_0^{\frac{1}{2}} x \cdot \left(\frac{1}{2} - x + \frac{8}{3}x^3\right) dx = \frac{21}{160},$$

it follows from Definition 3.1 that

$$\tau_p(B|A) = \frac{\tau_p(B \otimes A^2)}{\tau_p(A^2)} = \frac{7}{20}, \text{ and } \tau_p(A|B) = \frac{\tau_p(A \otimes B^2)}{\tau_p(B^2)} = \frac{63}{80},$$

Since

$$\tau_p\left((A \wedge B) \otimes A^2\right) = \tau_p\left((q_1 \wedge q_2) \otimes {q_1}^2\right) = \int_0^1 (x+1) \cdot \left(\frac{1}{3}x^3 - x^2 + x\right) dx = \frac{7}{120},$$

it follows from Definition 3.1 that

$$\tau_p(A \wedge B|A) = \frac{\tau_p\left((A \wedge B) \otimes A^2\right)}{\tau_p(A^2)} = \frac{1}{10},$$

Since

$$\tau_p(A \wedge B|\Sigma) = \tau_p(A \wedge B|A) = \frac{1}{10},$$

According to Theorem 5.2, we have

$$div_\Sigma(\Gamma|\Sigma) = 1 - \tau_p(A \wedge B|\Sigma) = \frac{9}{10}.$$

Theorem 5.3. In the BL^* system, let p be a sequence of valuation density functions, $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\otimes \Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes \Sigma^2) > 0$, Γ is a theory in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ. If Γ is an inconsistent theory under the condition Σ, then $div_\Sigma(\Gamma|\Sigma) = 1$, i.e., an inconsistent theory under the condition Σ is $\Sigma-$conditionally randomly fully divergent.

Proof. Let Γ be an inconsistent theory under the condition Σ. Suppose Γ is a theorem and $\bar{0}$ is a contradiction. From Definition 4.2, we can obtain $d_\Sigma(\Gamma, \bar{0}|\Sigma) = 1$.

Consequently, from Definition 5.1, the divergence degree of Γ under the condition Σ, denoted as $div_\Sigma(\Gamma|\Sigma) = sup\{d_\Sigma(A, B|\Sigma)|A, B \in D(\Gamma)\} = 1$. This means that an inconsistent theory under the condition Σ is $\Sigma-$conditionally randomly fully divergent.

With the concepts of $\Sigma-$conditional random similarity, $\Sigma-$conditional random pseudo-metric, and $\Sigma-$conditional random divergence established, we now present the theory of approximate reasoning based on condition Σ in $F(S)$.

Definition 5.2. In the BL^* system, let p be a sequence of valuation density functions, $A, B \in F(S)$, $\Sigma = \{A_1, A_2, \cdots, A_n\} \subset F(S)$, $\otimes \Sigma^2 = {A_1}^2 \otimes {A_2}^2 \otimes \cdots {A_n}^2$, $\tau_p(\otimes \Sigma^2) > 0$, Γ and Γ' are theories in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ, $D(\Gamma')$ denotes the set of all consequences of theory Γ', $\forall \varepsilon > 0$.

(1) If $inf\{d_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}<\varepsilon$, then A is called a consequence of Γ with I −type error less than ε under condition Σ, denoted as $A\in D^{1}_{\Sigma,\varepsilon}(\Gamma)$;

(2) If $1-sup\{\tau_p(B\to A|\Sigma)|B\in D(\Gamma)\}<\varepsilon$, then A is called a consequence of Γ with II −type error less than ε under condition Σ, denoted as $A\in D^{2}_{\Sigma,\varepsilon}(\Gamma)$;

(3) If $inf\left\{H\left(D(\Gamma),D(\Gamma^{'})\right)\middle|\Gamma^{'}\subseteq F(S),\Gamma^{'}\nvdash A\right\}<\varepsilon$, then A is called a consequence of Γ with III −type error less than ε under condition Σ, denoted as $A\in D^{3}_{\Sigma,\varepsilon}(\Gamma)$, Here H denotes the Hausdorff distance.

Theorem 5.4. In the BL^{*} system, let p be a sequence of valuation density functions, $A,B\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$, $\tau_p(\otimes\,\Sigma^2)>0$, Γ is a theory in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ, $\forall\varepsilon>0$. If$A\in D^{1}_{\Sigma,\varepsilon}(\Gamma)$,then $A\in D^{2}_{\Sigma,\varepsilon}(\Gamma)$.

Proof. If $A\in D^{1}_{\Sigma,\varepsilon}(\Gamma)$, then $inf\{d_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}<\varepsilon$. According to Definition 4.2, we have

$$inf\{d_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}=inf\{1-\xi_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}=1-sup\{\xi_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}.$$

By Definition 4.1, it follows that

$$\xi_{\Sigma}(A,B|\Sigma)=\tau_p\big((A\to B)\wedge(B\to A)|\Sigma\big)\le\tau_p(B\to A|\Sigma).$$

Therefore,

$$1-sup\{\tau_p(B\to A|\Sigma)|B\in D(\Gamma)\}\le 1-sup\{\xi_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}=inf\{d_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}<\varepsilon,$$

and hence $A\in D^{2}_{\Sigma,\varepsilon}(\Gamma)$.

Theorem 5.5. In the BL^{*} system, let p be a sequence of valuation density functions, $A,B\in F(S)$, $\Sigma=\{A_1,A_2,\cdots,A_n\}\subset F(S)$,$\otimes\,\Sigma^2={A_1}^2\otimes{A_2}^2\otimes\cdots{A_n}^2$, $\tau_p(\otimes\,\Sigma^2)>0$, Γ and $\Gamma^{'}$ are theories in $F(S)$, and $D(\Gamma)$ denotes the set of all consequences of theory Γ, $D(\Gamma^{'})$ denotes the set of all consequences of theory $\Gamma^{'}$, $\forall\varepsilon>0$. If$A\in D^{3}_{\Sigma,\varepsilon}(\Gamma)$,then $A\in D^{1}_{\Sigma,\varepsilon}(\Gamma)$ and $A\in D^{2}_{\Sigma,\varepsilon}(\Gamma)$.

Proof. If $A\in D^{3}_{\Sigma,\varepsilon}(\Gamma)$, then according to Definition 5.2 (3), there exists $\Gamma^{'}\subset F(S)$ such that $\Gamma^{'}\nvdash A$, and $H\left(D(\Gamma),D(\Gamma^{'})\right)<\varepsilon$. In this case, $A\in D(\Gamma^{'})$, so

$$inf\{d_{\Sigma}(A,B|\Sigma)|B\in D(\Gamma)\}\le d_{\Sigma}(A,D(\Gamma)|\Sigma)\le H\left(D(\Gamma),D(\Gamma^{'})\right)<\varepsilon.$$

By Definition 5.2 (1), we have $A\in D^{1}_{\Sigma,\varepsilon}(\Gamma)$, and then by Theorem 5.4, it follows that $A\in D^{2}_{\Sigma,\varepsilon}(\Gamma)$.

The conditional randomized truth degree can provide a precise mathematical model for uncertain reasoning. One of the core challenges in artificial intelligence is handling fuzzy, random, and conditionally constrained reasoning in real-world scenarios. The conditional randomized truth degree offers a quantifiable and computable logical tool for this purpose, representing its most critical application domain. In classic scenarios of fuzzy reasoning, such as Fuzzy Modus Ponens, Fuzzy Hypothetical Syllogism, and multi-rule fuzzy reasoning, the conditional randomized truth degree enables quantitative assignment and computation of the “credibility of premise propositions” and the “credibility of reasoning conclusions”. It clarifies the “random credibility of the reasoning conclusion being true under given premise conditions”, addressing the limitation of traditional fuzzy reasoning, which “can only conduct qualitative reasoning without quantifying credibility”.

6. Conclusion and Future Work

In the fuzzy propositional logic system BL^{*} with an assignment domain of [0,1], this paper introduces the concepts of conditional randomized truth degree, conditional randomized similarity degree, and conditional randomized pseudo-metric for propositional formulas. It establishes a conditional randomized logical metric space and further introduces the concept of conditional randomized divergence for theories within this space. These studies enable the quantification of the truth degree of propositions under specific conditions when dealing with uncertain information characterized by randomness and premise constraints, providing a basis for assessing the credibility of information and more accurately processing fuzzy or random input information. The paper also proposes three different types of approximate reasoning modes under condition Σ and discusses their relationships, offering quantitative tools for approximate reasoning and effective fuzzy reasoning methods for fuzzy control applications. In our future endeavors, we aim to extend the theories pertaining to conditional randomized truth degrees and conditional randomized divergence to other logical systems. Furthermore, we will delve into an in-depth exploration of the conditional randomized consistency within the fuzzy propositional logic systems BL^*, along with investigating the practical applications of these theories in domains such as artificial intelligence and fuzzy reasoning.

Conflict of Interest

The authors declare that there are no conflicts of interest.

Funding

This research was supported by the National Natural Science Foundation of China under grant numbers 12261090,12301456.

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